“Output Regulation of Nonlinear Systems”

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Outline

- Center Manifold Theory
- Problem formulation
- Linear theory review
- Assumptions for nonlinear regulator problems
- Solution of nonlinear regulator problems
- Nonlinear zero dynamics
- Solvability of regulator problem in terms of zero dynamics
- Application

- Consider

$$\dot{x} = f(x) \tag{A.1}$$

where f is a C^r manifold ($r \geq 2$), $x \in X \subset \mathbb{R}^n$, and $x = 0$ is eq. pt.

- $F := \frac{\partial f}{\partial x}(0)$ can be decomposed into three eigenspaces,
  - E^+ corr. to (n^+) eigenvalues in ORHP
  - E^- corr. to (n^-) eigenvalues in OLHP
  - E° corr. to (n°) eigenvalues on imaginary axis

- Suppose that $n^+ = 0$ and $n^\circ \neq 0$. (A.1) can be transformed into

$$\dot{y} = Ay + g(y, z) \tag{A.2a}$$
$$\dot{z} = Bz + h(y, z) \tag{A.2b}$$

where

  - $A \in \mathbb{R}^{n^- \times n^-}$ is Hurwitz
  - $B \in \mathbb{R}^{n^\circ \times n^\circ}$ has all e-values on $\jmath\omega$-axis
  - h, g vanish at $(0, 0)$ along with (first-order) derivatives

Is system stable? Depends on its behavior on *Center Manifold*

Defn: A submanifold S of X is *locally invariant* for (A.1) if for each $x^\circ \in S$ there exist $t_1 < 0 < t_2$ such that the integral curve of (A.1) satisfying $x(0) = x^\circ$ has $x(t) \in S$ for all $t \in (t_1, t_2)$.

Defn: A *center manifold for (A.1) at* $x = 0$ is a locally invariant manifold S passing through $x = 0$ with $T_0 S = E^\circ$.

Center Manifold Theorem: Existence and Characterization

> *Thm (Carr [3]):* For (A.2), $\exists$ a nbhd $V \subset \mathbb{R}^{n^\circ}$ of $z = 0$ and a C^{r-1} mapping $\pi : V \to \mathbb{R}^{n^-}$ such that
>
> $$S := \{(y, z) \in \mathbb{R}^{n^-} \times V : y = \pi(z)\}$$
>
> is a center manifold.

- By definition, a center manifold of (A.2) at $(0, 0)$ must pass through $(0, 0)$ and be tangent to $\{(y, z) : y = 0\}$, hence

$$\pi(0) = 0, \qquad \frac{\partial \pi}{\partial z}(0) = 0 \tag{A.3a}$$

- Furthermore, it is by defn locally invariant, hence

$$\frac{\partial \pi}{\partial z}(Bz + h(\pi(z), z)) = A\pi(z) + g(\pi(z), z) \tag{A.3b}$$

- Summary: A center manifold always exists and can be characterized as the graph of a function π that satisfies the PDE (A.3b) subject to the constraints of (A.3a).

- Note: A center manifold need not be unique. Also, a C^∞ system may not have a C^∞ center manifold, but will always have a C^k manifold for any $2 \le k < \infty$.

Thm (Carr [3]): Suppose that $y = \pi(z)$ represents a center manifold of (A.2) at (0,0), and let $(y(t), z(t))$ be a trajectory of (A.2). There exists a nbhd U° of $(0,0)$ and $M, a > 0$ such that if $(y(0), z(0)) \in U^\circ$ then

$$\|y(t) - \pi(z(t))\| \leq Me^{-at}\|y(0) - \pi(z(0))\|$$

for all $t \geq 0$ as long as $(y(t), z(t)) \in U^\circ$.

- In other words, the center manifold is locally attractive.

Reduction Principle (Carr [3]): If the reduced system

$$\dot{\zeta} = B\zeta + h(\pi(\zeta), \zeta)$$

has a stable (resp., a.s., unstable) eq. pt. at $\zeta = 0$, then the original system (A.2) has a stable (resp., a.s., unstable) eq. pt. at $(y, z) = (0, 0)$.

- In other words, the reduced system on the center manifold determines the stability of the entire system.

- Consider the tracking/regulator problem

$$\begin{array}{lrcl} \text{system:} & \dot{x} & = & f(x) + g(x)u + p(x)w \\ \text{exosystem:} & \dot{w} & = & s(w) \\ \text{error:} & e & = & h(x) + q(w) \end{array}$$

  $x \in X \subset I\!R^n$ is state, $u \in I\!R^m$ is input, $w \in W \subset I\!R^s$ is reference or disturbance, $e \in I\!R^p$ is error. All v.f. are smooth, and $f(0) = 0$, $s(0) = 0$, $h(0) = q(0) = 0$.

- Want to regulate system: design controller so that

  1. closed loop system is exp. s. when $w \equiv 0$

  2. $e(t) \to 0$.

  This captures both disturbance attenuation ($q(w) \equiv 0$) and tracking.

- *State feedback controller*: $u = \alpha(x, w)$
  (assume $\alpha(0, 0) = 0$)

- *Error feedback controller*:

$$\begin{array}{rcl} \dot{z} & = & \eta(z, e) \\ u & = & \theta(z) \end{array}$$

  with $z \in Z \subset I\!R^\nu$. (assume $\eta(0, 0) = 0$, $\theta(0) = 0$)

Problem 1: State Feedback Regulator Problem
Find $\alpha(x,w)$ such that

1a) the equilibrium $x=0$ of

$$\dot{x} = f(x) + g(x)\alpha(x,0)$$

is exponentially stable.

1b) $\exists$ a nbhd $U \subset X \times W$ of $(0,0)$ such that for each $(x(0), w(0)) \in U$, the closed loop system

$$\dot{x} = f(x) + g(x)\alpha(x,w) + p(x)w \tag{2.4a}$$
$$\dot{w} = s(w) \tag{2.4b}$$

has solution with

$$\lim_{t\to\infty} h(x(t)) + q(w(t)) = 0.$$

Problem 2: Error Feedback Regulator Problem
Find $\eta(z,e)$ and $\theta(z)$ such that

2a) the equilibrium $(0,0)$ of

$$\begin{aligned} \dot{x} &= f(x) + g(x)\theta(z) \\ \dot{z} &= \eta(z, h(x)) \end{aligned}$$

is exponentially stable.

2b) $\exists$ a nbhd $U \subset X \times Z \times W$ of $(0,0,0)$ such that for each $(x(0), z(0), w(0)) \in U$, the closed loop system

$$\dot{x} = f(x) + g(x)\theta(z) + p(x)w \tag{2.6a}$$
$$\dot{z} = \eta(z, h(x) + q(w)) \tag{2.6b}$$
$$\dot{w} = s(w) \tag{2.6c}$$

has solution with

$$\lim_{t\to\infty} h(x(t)) + q(w(t)) = 0.$$

Notation for Linear Approximations

- linear approx. of system

$$\dot{x} = Ax + Bu + Pw \tag{3.1a}$$

$$\dot{w} = Sw \tag{3.1b}$$

$$e = Cx + Qw \tag{3.2}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}, \quad B = g(0), \quad P = p(0),$$

$$S = \left.\frac{\partial s}{\partial w}\right|_{w=0}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{x=0}, \quad Q = \left.\frac{\partial q}{\partial w}\right|_{w=0}$$

- linear approx. of state feedback controller

$$u = Kx + Lw$$

where

$$K = \left.\frac{\partial \alpha}{\partial x}\right|_{(x,w)=(0,0)} \quad \text{and} \quad L = \left.\frac{\partial \alpha}{\partial w}\right|_{(x,w)=(0,0)}$$

- linear approx. of error feedback controller

$$\begin{aligned} \dot{z} &= Fz + Ge \\ u &= Hz \end{aligned}$$

where

$$F = \left.\frac{\partial \eta}{\partial z}\right|_{(z,e)=(0,0)}, \quad G = \left.\frac{\partial \eta}{\partial e}\right|_{(z,e)=(0,0)}, \quad H = \left.\frac{\partial \theta}{\partial w}\right|_{z=0}$$

Linear Regulator Theory: Assumptions

- For linear system as above, make following assumptions

A1: All eigenvalues of S are in the closed RHP.

A2: (A, B) is stabilizable.

A3: the pair

$$\left(\begin{bmatrix} C & Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$$

is detectable.

- *A1* is no loss of generality, since a.s. modes of exosystem do not affect asymptotic regulation.

- *A2* is necessary for attainment of exp. stability

- *A3* is really no more restrictive than detectability of (C, A): If (C, A) is detectable but *A3* does not hold, can reduce exosystem to portion which is detectable via error, since effect of w on e is all that matters.

Proposition 1 (from Francis [5]): Suppose *A1* and *A2* hold. Then the linear state feedback regulator problem is solvable iff $\exists$ matrices Π and Γ solving

$$A\Pi + B\Gamma = \Pi S \tag{3.4a}$$

$$C\Pi + Q = 0. \tag{3.4b}$$

Proposition 2 (from Francis [5]): Suppose *A1*, *A2*, and *A3* hold. Then the linear *error* feedback regulator problem is solvable iff the linear *state* feedback regulator problem is solvable.

Proposition 3 (from Hautus [7]): (3.4) are solvable iff the original system (3.1)-(3.2) and the system

$$\dot{x} = Ax + Bu \tag{3.6a}$$

$$\dot{w} = Sw \tag{3.6b}$$

$$e = Cx \tag{3.7}$$

have the same *transmission polynomials (?)*, with u as input and e as output.

This paper extends these results to nonlinear systems using the Center Manifold Theorem.

Nonlinear Regulator Problems: Assumptions

Defn: An initial condition w° of the nonlinear system

$$\dot{w} = s(w); \;\; w(0) = w^\circ$$

is *Poisson stable* if

1. the flow $\Phi_t^0(w^\circ)$ of the system is defined for all $t \in \mathbb{R}$
2. for each nbhd $U \ni w^\circ$ and for each $T > 0$ there exists a time $t_1 < -T$ such that $\Phi_{t_1}^0(w^\circ) \in U$ and there exists a time $t_2 > T$ such that $\Phi_{t_2}^0(w^\circ) \in U$.

For example, a point on a limit cycle is Poisson stable.

Assumptions

H1: $w = 0$ is a stable eq. pt. of the exosystem. Further, $\exists$ a nbhd $\widehat{W} \ni 0$ with the property that each i.c. $w(0) \in \widehat{W}$ is *Poisson* stable.

H2: The pair $f(x), g(x)$ has a stabilizable linear approximation at $x = 0$.

H3: The pair

$$\left(\begin{bmatrix} f(x) + p(x)w \\ s(w) \end{bmatrix}, \quad h(x) + q(w) \right)$$

has a detectable linear approx at $(x, w) = (0, 0)$.

- *H1* implies that S must have purely imaginary eigenvalues.
- *H1* is not too restrictive, since unstable trajectories are not of interest, and a.s. trajectories do not effect asymptotic error.
- General bounded signals are excluded by *H1*?

Lemma (Vidyasagar [16]): Consider system with triangular structure

$$\dot{x} = \phi(x, w)$$
$$\dot{w} = s(w)$$

where $\phi(0,0) = 0$ and $s(0) = 0$, both suff. smooth.

If $x = 0$ is an a.s. equilibrium of $\dot{x} = \phi(x, 0)$ and if $w = 0$ is a stable equilibrium of $\dot{w} = s(w)$, then $(x, w) = (0, 0)$ is a stable equilibrium of the entire system.

Using this Lemma and *H1*, requirement 1a) (resp., 2a) implies that the closed loop system including exosystem (2.4) (resp., (2.6)) is stable.

Lemma 1: Suppose *H1* holds and for some $\alpha(x, w)$, 1a) is satisfied. Then 1b) is also satisfied iff there exists a locally-defined C^k ($k \geq 2$) mapping $x = \pi(w)$ with $\pi(0) = 0$ and satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\alpha(\pi(w), w) + p(\pi(w))w \quad (5.1a)$$
$$0 = h(\pi(w)) + q(w) \quad (5.1b)$$

Theorem 1: Under *H1* and *H2*, the state feedback regulator problem is solvable iff there exist C^k ($k \geq 2$) mappings $x = \pi(w)$ and $u = c(w)$ with $0 = \pi(0)$ and $c(0) = 0$ both locally defined about the origin satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \quad (5.2a)$$

$$0 = h(\pi(w)) + q(w) \quad (5.2b)$$

- If the conditions (5.2) are met, then the desired controller is given by

$$\alpha(x, w) = c(w) + K(x - \pi(w)).$$

  where K makes $(A + BK)$ Hurwitz.

- (5.2a) means that the graph of $x = \pi(w)$ is rendered locally invariant by the feedback $u = c(w)$. (5.2b) means that this manifold is annihilated by the error map $e = h(x) + q(w)$.

- In the linear case, Theorem 1 reduces to Proposition 1, with $\pi(w) = \Pi w$ and $c(w) = \Gamma w$.

Lemma 2: Suppose *H1* holds and for some $\eta(z,e)$ and $\theta(z)$, 2a) is satisfied.

Then 2b) is also satisfied iff there exists a locally-defined C^k $(k \geq 2)$ mappings $x = \pi(w)$ and $z = \sigma(w)$ with $\pi(0) = 0$ and $\sigma(0) = 0$ that satisfy

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\theta(\sigma(w)) + p(\pi(w))w \quad (5.3a)$$

$$\frac{\partial \sigma}{\partial w} s(w) = \eta(\sigma(w), 0) \quad (5.3b)$$

$$0 = h(\pi(w)) + q(w) \quad (5.3c)$$

Here, the graph of $(x,z) = (\pi(w), \sigma(w))$ is the center manifold.

Theorem 2: Under *H1*, *H2*, and *H3*, the error feedback regulator problem is solvable iff there exist C^k $(k \geq 2)$ mappings $x = \pi(w)$ and $u = c(w)$ with $0 = \pi(0)$ and $c(0) = 0$ both locally defined about the origin satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \quad (5.2a)$$

$$0 = h(\pi(w)) + q(w) \quad (5.2b)$$

Note that (5.3b) expresses the fact that the controller incorporates an internal model of the exosystem.

Nonlinear Zero Dynamics

Defn: A submanifold M is *locally controlled invariant* for

$$\dot{x} = f(x) + g(x)u$$

if there exists $u^* : M \to \mathbb{R}^m$ smooth and a nbhd $U \ni 0$ ($U \subset \mathbb{R}^n$) such that the vector field $f(x)+g(x)u^*(x) \in T_xM$ for all $x \in M \cap U$.

Defn: An *output zeroing submanifold* for

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

is a connected submanifold $M \ni 0$ that satisfies
i) M is locally controlled invariant, and
ii) $h(x) = 0$ for all $x \in M$.
The output zeroing submanifold is called *locally maximal* if for some nbhd $U \ni 0$, $M \cap U \supset \bar{M} \cap U$ for all output zeroing submanifolds $\bar{M}$.

- Fairly technical conditions involving span$\{g_1(x), \ldots, g_m(x)\}$ are given for existence of a locally maximal output zeroing submanifold and for uniqueness of u^* (Proposition 4).

- If these conditions are met, then the vector field

$$f^*(x) := f(x) + g(x)u^*(x)$$

is tangent to the locally maximal output zeroing submanifold, Z^*. The pair (Z^*, f^*) are called the *zero dynamics* of the system.

- The system

$$\dot{x} = f^*(x); \;\; x \in Z^*$$

identifies the internal dynamics consistent with output being forced to zero by proper choice of input and initial condition.

- Let the original system/exosystem by redefined as

$$\Sigma_e: \begin{array}{rcl} \dot{x}_e & = & f_e(x_e) + g_e(x_e)u \\ e & = & h_e(x_e) \end{array}$$

where $x_e := \text{col}(x, w) \in X_e = X \times W$.

- Define the system itself, without the exosystem, as

$$\Sigma: \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ e & = & h(x). \end{array}$$

Here $x \in X$, and u, e are as in Σ_e.

- If (5.2) are satisfied, the submanifold

$$M_s := \{(x, w) \in X_e : x = \pi(w)\}$$

is an output zeroing submanifold of the system Σ_e, since (5.2a) means that M_s is locally controlled invariant (with $u^*(x_e) = c(w)$), and (5.2b) means that it is annihilated by the output map h_e.

Assumptions

Z1: the system Σ has zero dynamics (Z^*, f^*) defined in a nbhd of the origin, and the feedback law that achieves invariance, u^* is unique.

Z2: the system Σ_e has zero dynamics (Z_e^*, f_e^*) defined in a nbhd of the origin, and the feedback law that achieves invariance, u_e^* is unique.

- There may be *another* output zeroing submanifold. Consider the zero dynamics of Σ, and let u^* be the (unique) feedback that renders $f^*(x) = f(x) + g(x)u^*(x)$ invariant to Z^*. Then the submanifold

$$M_z = Z^* \times \{0\} \tag{7.2}$$

of X_e is an output zeroing submanifold of Σ_e. That is, it is invariance is achieved under u^* since $w = 0$ is an equilibrium of $\dot{w} = s(w)$, and is output zeroing since $h(x) = 0$ for $x \in Z^*$ and $q(0) = 0$.

Lemma 3: Suppose that *Z1* and *Z2* hold, and that the set

$$M = Z_e^* \cap (X \times \{0\}) \tag{7.3}$$

is a smooth manifold in a nbhd of $x_e = 0$. Then M locally coincides with M_z defined in (7.2). Moreover, M is locally invariant under f_e^* and $f_e^*|_M$ is locally diffeomorphic to the v.f. f^* which characterizes the zero dynamics of Σ.

Theorem 3: Suppose *Z1* and *Z2* hold. There exist smooth mappings $x = \pi(w)$ and $u = c(w)$ with $0 = \pi(0)$ and $c(0) = 0$ both locally defined about the origin satisfying conditions (5.2) iff the zero dynamics (Z_e^*, f_e^*) of Σ_e satisfy

i) M defined by (7.3) is a smooth manifold in a nbhd of x_e.

ii) There exists a submanifold Z_s of Z_e^*, of dimension s with $Z_s \ni 0$ and

$$T_0 Z_e^* = T_0 Z_s \oplus T_0 M$$

iii) Z_s is locally invariant under f_e^*, and $f_e^*|_{Z_s}$ is locally diffeomorphic to the vector field $s(w)$ which characterizes the exosystem.

- By Lemma 3, i) implies that M is locally invariant under f_e^* and that $f_e^*|_M$ is locally diffeomorphic to f^*.

- Hence (5.2) are solvable iff the zero dynamics of the entire system Σ_e possess two complimentary invariant submanifolds, with the flows of f_e^* on these two submanifolds being (to within a diffeomorphism) that of the exosystem and that of the zero dynamics of the system Σ.

- The proof of the Theorem illustrates that if Σ_e is known, along with u_e^*, the problem reduces to finding a submanifold Z_s of Z_e^* with the properties
  - dim Z_s is s;
  - Z_s is transverse to $X \times \{0\}$;
  - Z_s is locally invariant under f_e^*; and
  - $f_e^*|_{Z_s}$ is equivalent under diffeomorphism to $s(w)$.

Corollary 1: Suppose *Z1* and *Z2* hold, along with *H1*, *H2*, and *H3*. Then either the state or error feedback regulator problem is solvable if

- $T_0 Z_e^* + T_0(X \times \{0\}) = T_0 X_e$; and
- the zero dynamics of Σ have a hyperbolic equilibrium at $x = 0$.

- Consider a SISO system with $p(x) = 0$, *i.e.*, it is only desired to track a reference output $y_R(t) = -q(w(t))$.
- Suppose that the system Σ has relative degree r. Clearly, the entire system Σ_e has relative degree r also.
- Applying the input-output exact linearization change of coordinates for Σ to Σ_e yields

$$\begin{aligned}
\dot{z} &= f_0(z, \xi) \\
\dot{w} &= s(w) \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= b(z, \xi) + a(z, \xi)u \\
e &= \xi_1 + q(w)
\end{aligned}$$

- The zero dynamics of Σ is given by

$$\dot{z} = f_0(z, 0)$$

- To obtain the zero dynamics of Σ_e, set $e \equiv 0$, and differentiate to obtain

$$\xi_k = -L_{s(w)}^{k-1} q(w)$$

for all $k \in \{1, \ldots, r\}$. Hence the zero dynamics of Σ_e are characterized by

$$\dot{z} = f_0(z, k(w)) \tag{8.0a}$$

$$\dot{w} = s(w) \tag{8.0b}$$

where $k(w) := -\text{col}(q(w), L_s q(w), \ldots, L_s^{r-1} q(w))$, and u_e^* is given by

$$u_e^*(x, w) = -\frac{L_f^r h(x) + L_s^r q(w)}{L_g L_f^{r-1} h(x)}$$

- Clearly *Z1* and *Z2* are satisfied.
- In this case, M in (7.3) is locally the integral submanifold of (8.0) with $w \equiv 0$. Hence $M = M_z$ as asserted by *Lemma 3*.
- Note

$$T_0 Z_e^* = \text{span}\{\text{col}(0, f_0, s(w))\}$$

and

$$T_0(X \times \{0\}) = \text{span}\{\text{col}(1, 1, 0)\}$$

hence condition i) of *Corollary 1* is satisfied.

- Hence, the problem is solvable if

$$\dot{z} = f_0(z, 0)$$

has a hyperbolic equilibrium at the origin.

- Note: conditions ii) and iii) of *Theorem 3* are satisfied iff $\exists$ a mapping $z = \lambda(x)$ whose graph is a center manifold of (8.0).